

A SPECTRAL SEQUENCE CONNECTING CONTINUOUS WITH LOCALLY CONTINUOUS GROUP COHOMOLOGY

MARTIN FUCHSSTEINER

ABSTRACT. We present a spectral sequence connecting the continuous and 'locally continuous' group cohomologies for topological groups. As an application it is shown that for contractible topological groups these cohomology concepts coincide.

INTRODUCTION

There exist various cohomology concepts for topological groups G and topological coefficient groups V which take the topologies of the group and that of the coefficients into account. One is obtained by restricting oneself to the complex $C_c^*(G; V)$ continuous group cochains only whose cohomology is called the *continuous group cohomology* $H_c(G; V)$. For abstract groups G and G -modules V the first cohomology group $H^1(G; V)$ classifies crossed morphisms modulo principal derivations, the second cohomology group $H^2(G; V)$ classifies equivalence classes of group extensions $V \hookrightarrow \hat{G} \rightarrow G$ and the third cohomology group $H^3(G; V)$ classifies equivalence classes crossed modules with kernel V and cokernel G (cf. [Wei94, Theorem 6.4.5, Theorem 6.6.3 and Theorem 6.6.13]). Analogous considerations show that for topological groups G and G -modules V the first cohomology group $H_c^1(G; V)$ classifies continuous crossed morphisms modulo principal derivations, the second cohomology group $H_c^2(G; V)$ classifies equivalence classes of topological group extensions $V \hookrightarrow \hat{G} \rightarrow G$ which admit a global section (i.e. $\hat{G} \rightarrow G$ is a trivial V -principal bundle) and the third cohomology group $H_c^3(G; V)$ classifies equivalence classes of topologically split crossed modules. The continuous group cohomology has the drawback that for even the compact Hausdorff group $G = \mathbb{R}/\mathbb{Z}$ the short exact sequence

$$0 \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$$

of coefficients does not induce a long exact sequence of cohomology groups. (The group $H_c^1(G; \mathbb{R})$ is trivial because the projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ does not admit global sections, $H_c^n(G; \mathbb{Z}) = 0$ because all continuous group cochains on G are constant whereas the group of $H_c^1(G; G)$ all continuous endomorphisms of G is non-trivial.)

This drawback is relieved by a second more general cohomology concept, which is obtained by considering the complex $C_{cg}^*(G; V)$ of group cochains which are continuous on some identity neighbourhood in G . By abuse of language some people call the corresponding cohomology groups $H_{cg}(G; V)$ the 'locally continuous group cohomology'. The first cohomology group $H_{cg}^1(G; V)$ classifies continuous crossed morphisms modulo principal derivations, the second cohomology group $H_{cg}^2(G; V)$

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classifies equivalence classes of topological group extensions $V \hookrightarrow \hat{G} \twoheadrightarrow G$ which admit local sections (i.e. $\hat{G} \twoheadrightarrow G$ is a locally trivial V -principal bundle) and the third cohomology group $H_{cg}^3(G; V)$ classifies equivalence classes of crossed modules in which all homomorphisms admit local sections.

The inclusion $C_{cg}^*(G; V) \hookrightarrow C_c^*(G; V)$ of cochain complexes induces a morphism $H_{cg}^*(G; V) \rightarrow H_c^*(G; V)$ of cohomology groups, which is used to compare the two cohomology concepts. As the above example shows, these cohomology concepts do not even coincide for connected compact Hausdorff groups and real coefficients. In the following we will show that the contractibility of a topological group G forces the two cohomologies to coincide (cf. Corollary 3.17):

Theorem. *For contractible groups G the inclusion $C_c^*(G; V) \hookrightarrow C_{cg}^*(G; V)$ induces an isomorphism in cohomology.*

This is proved by constructing a row-exact double complex $A_{cg,eq}(G; V)$ whose rows and columns can be augmented by the complexes $C_{cg}^*(G; V)$ and $C_c^*(G; V)$ respectively. The contractibility of G will be shown to force the columns of this double complex to be exact as well, which then in turn is shown to imply that the inclusion $C_{cg}^*(G; V) \hookrightarrow C_c^*(G; V)$ induces an isomorphism in cohomology. In fact we will be considering the more general setting of transformation groups (G, X) and G -equivariant cochains on X and prove these results in this more general setting. Similar results for k -groups and smooth transformation groups will also be obtained.

1. BASIC CONCEPTS

In this section we recall the definitions of various cochain complexes and the interpretation of some of their cohomology groups. For topological spaces X and abelian topological groups V one can consider variations of the exact *standard complex* $A^*(X; V) = \text{hom}_{\text{Set}}(X; V)$ of abelian groups.

Definition 1.1. For every topological space X and abelian topological group V the subcomplex $A_c^*(X; V) := C(X^{*+1}; V)$ of the standard complex is called the *continuous standard complex*.

For transformation groups (G, X) and G -modules V the group G acts on the spaces X^{n+1} via the diagonal action and the groups $A^n(X; V)$ can be endowed with a G -action via

$$(1.1) \quad G \times A^n(X; V) \rightarrow A^n(X; V), \quad [g.f](\vec{x}) = g.[f(g^{-1}.\vec{x})].$$

The G -fixed points of this action are the G -equivariant cochains. Because the differential of the standard complex intertwines the G -action, the equivariant cochains form a subcomplex $A^*(X; V)^G$ of the standard complex and the continuous equivariant cochains form a subcomplex $A_c^*(X; V)^G$ of the continuous standard complex. These complexes are not exact in general.

Example 1.2. For any group G which acts on itself by left translation and G -module V the complex $A^*(G; V)^G$ is the complex of (homogeneous) group cochains; for topological groups G and G -modules V the complex $A_c^*(G; V)^G$ is the complex of continuous (homogeneous) group cochains

Definition 1.3. The cohomology $H_{eq}(X; V)$ of the complex $A^*(X; V)^G$ is called the equivariant cohomology of X (with values in V). The cohomology $H_{eq,c}(X; V)$

of the subcomplex $A_c^*(X; V)^G$ is called the equivariant continuous cohomology of X (with values in V).

Example 1.4. For any group G which acts on itself by left translation and G -module V the cohomology $H_{eq}(G; V)$ is the group cohomology of G with values in V ; for topological groups G and G -modules V the cohomology $H_{eq,c}(G; V)$ is the continuous group cohomology of G with values in V .

For transformation groups (G, X) and G -modules V there exists a G -invariant complex $A_{cg}^*(X; V)$ between $A_c(X; V)$ and $A(X; V)$ which we are going to define now. For each open covering \mathfrak{U} of X and each $n \in \mathbb{N}$ one can define an open neighbourhood $\mathfrak{U}[n]$ of the diagonal in X^{n+1} via

$$\mathfrak{U}[n] := \bigcup_{U \in \mathfrak{U}} U^{n+1}.$$

These neighbourhoods of the diagonals in X^{*+1} form a simplicial subspace of X^{*+1} which allows us to consider the subcomplex of $A^*(X; V)$ formed by the groups

$$A_{cr}^n(X, \mathfrak{U}; V) := \{f \in A^n(X; V) \mid f|_{\mathfrak{U}[n]} \in C(\mathfrak{U}[n]; V)\}$$

of cochains whose restriction to the subspaces $\mathfrak{U}[n]$ of X^{n+1} are continuous. The cohomology of the cochain complex $A_{cr}^*(X, \mathfrak{U}; V)$ is denoted by $H_{cr}(X, \mathfrak{U}; V)$. If the covering \mathfrak{U} of X is G -invariant, then the subspaces $\mathfrak{U}[*]$ is a simplicial G -subspace of the simplicial G -space X^{*+1} .

Example 1.5. If $G = X$ is a topological group which acts on itself by left translation and U an open identity neighbourhood, then $\mathfrak{U}_U := \{g.U \mid g \in G\}$ is a G -invariant open covering of G and $\mathfrak{U}[*]$ is an open simplicial G -subspace of G^{*+1} .

For G -invariant coverings \mathfrak{U} of X the cohomology of the subcomplex $A_{cr}^*(X, \mathfrak{U}; V)^G$ of G -equivariant cochains is denoted by $H_{cr,eq}(X, \mathfrak{U}; V)$.

Example 1.6. If $G = X$ is a topological group which acts on itself by left translation and U an open identity neighbourhood, then the complex $A_{cr}^*(X, \mathfrak{U}_U; V)^G$ is the complex of homogeneous group cochains whose restrictions to the subspaces $\mathfrak{U}_U[*]$ are continuous. (These are sometimes called \mathfrak{U} -continuous cochains.)

For directed systems $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of X one can also consider the colimit complex $\text{colim}_i A_{cr}^*(X, \mathfrak{U}_i; V)$. In particular for the directed system of all open coverings of X one observes that the open diagonal neighbourhoods $\mathfrak{U}[n]$ in X^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods, hence one obtains the complex

$$A_{cg}^*(X; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } X} A_{cr}^*(X; \mathfrak{U}; V)$$

of global cochains whose germs at the diagonal are continuous. This is a subcomplex of the standard complex $A^*(X; V)$ which is invariant under the G -action (Eq. 1.1) and thus a sub complex of G -modules. The G -equivariant cochains with continuous germ form a subcomplex $A_{cg}^*(X; V)^G$ thereof, whose cohomology is denoted by $H_{cg,eq}(X; V)$. The latter subcomplex can also be obtained by taking the colimit over all G -invariant open coverings of X only:

Proposition 1.7. *The natural morphism of cochain complexes*

$$A_{cg,eq}^*(X; V) := \text{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } X} A_{cr}^*(X; \mathfrak{U}; V)^G \rightarrow A_{cg}^*(X; V)^G$$

is a natural isomorphism.

Proof. We show that this morphism is surjective and injective. Every equivalence class in $A_{cg}^n(X; V)^G$ can be represented by a cochain $f \in A_{cr}^n(X, \mathfrak{U}; V)^G$, where \mathfrak{U} is an open cover of X . The cochain f is continuous on $\mathfrak{U}[n]$ by definition. Its equivariance implies, that it also is continuous on $G.\mathfrak{U}[n] = (G.\mathfrak{U})[n]$, hence an element of $A_{eq}^n(X, (G.\mathfrak{U})[n]; V)$. The equivalence class $[f] \in A_{cg,eq}^n(X; V)$ is mapped onto $[f]A_{cg}^n(X; V)^G$. This proves surjectivity.

Every equivalence class in $A_{cg,eq}^n(X; V)^G$ can be represented by an equivariant n -cochain f in $A_{cr}^n(X, \mathfrak{U}; V)^G$, where \mathfrak{U} is a G -invariant open cover of X . If the image of the class $[f] \in A_{cg}^*(X; V)^G$ is trivial, then the cochain f itself is trivial and so is its class $[f] \in A_{cg,eq}^n(X; V)^G$. This proves injectivity. \square

Corollary 1.8. *The cohomology $H_{cg,eq}(X; V)$ is the cohomology of the complex of equivariant cochains which are continuous on some G -invariant neighbourhood of the diagonal.*

Example 1.9. If $G = X$ is a topological group which acts on itself by left translation, then the complex $A_{cg}^*(G; V)^G$ is the complex of homogeneous group cochains whose germs at the diagonal are continuous. (By abuse of language these are sometimes called 'locally continuous' group cochains.)

2. THE SPECTRAL SEQUENCE

Let (G, X) be a transformation group, V be G -module and \mathfrak{U} be an open covering of X . We will show (in Section 3) that the inclusion $A_{cr}^*(X, \mathfrak{U}; V) \hookrightarrow A_c^*(X; V)$ induces an isomorphism in cohomology provided the space X is contractible. For this purpose we consider the abelian groups

$$(2.1) \quad A_{cr}^{p,q}(X, \mathfrak{U}; V) := \{f : X^{p+1} \times X^{q+1} \rightarrow V \mid f|_{X^{p+1} \times \mathfrak{U}[q]} \text{ is continuous} \}.$$

The abelian groups $A_{cr}^{p,q}(X, \mathfrak{U}; V)$ form a first quadrant double complex whose vertical and horizontal differentials are given by

$$\begin{aligned} d_h^{p,q} : A_{cr}^{p,q} &\rightarrow A_{cr}^{p+1,q}, & d_h^{p,q}(f^{p,q})(\vec{x}, \vec{x}') &= \sum_{i=0}^{p+1} (-1)^i f^{p,q}(x_0, \dots, \widehat{x}_i, \dots, x_{p+1}, \vec{x}') \\ d_v^{p,q} : A_{cr}^{p,q} &\rightarrow A_{cr}^{p,q+1}, & d_v^{p,q}(f^{p,q})(\vec{x}, \vec{x}') &= (-1)^p \sum_{i=0}^{q+1} (-1)^i f^{p,q}(\vec{x}, x'_0, \dots, \widehat{x}'_i, \dots, x'_{q+1}). \end{aligned}$$

The double complex $A_{cr}^{*,*}(X, \mathfrak{U}; V)$ can be filtrated column-wise to obtain a spectral sequence $E_{cr,*}^{*,*}(X, \mathfrak{U}; V)$ (cf. [McC01, Theorem 2.15]). Since the double complex is a first quadrant double complex, the spectral sequence $E_{cr,*}^{*,*}(X, \mathfrak{U}; V)$ converges to the cohomology of the total complex of $A_{cr}^{*,*}(X, \mathfrak{U}; V)$.

The rows of the double complex $A_{cr}^{*,*}(X, \mathfrak{U}; V)$ can be augmented by the complex $A_{cr}^*(X, \mathfrak{U}; V)$ for the covering \mathfrak{U} and the columns can be augmented by the exact

complex $A_c^*(X; V)$ of continuous cochains:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
 A_{cr}^2(X, \mathfrak{U}; V) & \longrightarrow & A_{cr}^{0,2}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{1,2}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{2,2}(X, \mathfrak{U}; V) \xrightarrow{d_h} \cdots \\
 \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
 A_{cr}^1(X, \mathfrak{U}; V) & \longrightarrow & A_{cr}^{0,1}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{1,1}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{2,1}(X, \mathfrak{U}; V) \xrightarrow{d_h} \cdots \\
 \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
 A_{cr}^0(X, \mathfrak{U}; V) & \longrightarrow & A_{cr}^{0,0}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{1,0}(X, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{cr}^{2,0}(X, \mathfrak{U}; V) \xrightarrow{d_h} \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & A_c^0(X; V) & \xrightarrow{d_h} & A_c^1(X; V) & \xrightarrow{d_h} & A_c^2(X; V) \xrightarrow{d_h} \cdots
 \end{array}$$

We denote the total complex of the double complex $A_{cr}^{*,*}(X, \mathfrak{U}; V)$ by $\text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)$. The augmentations of the rows and columns of this double complex induce morphisms $i^* : A_{cr}^*(X, \mathfrak{U}; V) \rightarrow \text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)$ and $j^* : A_c^*(X; V) \rightarrow \text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)$ of cochain complexes respectively.

Lemma 2.1. *The morphism $i^* : A_{cr}^*(X, \mathfrak{U}; V) \rightarrow \text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology.*

Proof. On each augmented row $A_{cr}^q(X, \mathfrak{U}; V) \hookrightarrow A_{cr}^{*,q}(X, \mathfrak{U}; V)$ one can define a contraction $h^{*,q}$ via

$$(2.2) \quad h^{p,q} : A_{cr}^{p,q}(X, \mathfrak{U}; V) \rightarrow A_{cr}^{p-1,q}(X, \mathfrak{U}; V), \quad h^{p,q}(f)(\vec{x}, \vec{x}') = f(x_0, \dots, x_{p-1}, x'_0, \vec{x}').$$

Therefore the augmented rows are exact and the augmentation i^* induces an isomorphism in cohomology. \square

Remark 2.2. Note that for non-trivial \mathfrak{U} this construction does not work for the column complexes, because the so constructed cochains would not fulfil the continuity condition in Def. 2.1.

For G -invariant open coverings \mathfrak{U} of X one can consider the sub double complex $A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$ of $A_{cr}^{*,*}(X, \mathfrak{U}; V)$ whose rows are augmented by the cochain complex $A_{cr}^*(X, \mathfrak{U}; V)^G$ for the covering \mathfrak{U} and the columns can be augmented by the complex $A_c^*(X; V)^G$ of continuous equivariant cochains (which is not exact in general).

Lemma 2.3. *For G -invariant coverings \mathfrak{U} of X the morphism $i_{eq}^* := i^{*G}$ induces an isomorphism in cohomology.*

Proof. The contraction $h_{*,q}$ of the augmented rows $A_{cr}^q(X, \mathfrak{U}; V) \hookrightarrow \text{Tot}A_{cr}^{*,q}(X, \mathfrak{U}; V)$ defined in Eq. 2.2 is G -equivariant and thus restricts to a row contraction of the augmented sub-row $A_{cr}^q(X, \mathfrak{U}; V)^G \hookrightarrow \text{Tot}A_{cr}^{*,q}(X, \mathfrak{U}; V)^G$. \square

So the morphism $H(i_{eq}) : H_{cr,eq}(X, \mathfrak{U}; V) \rightarrow H(\text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)^G)$ is invertible. For the composition $H(i_{eq})^{-1}H(j_{eq}) : H_{c,eq}(X; V) \rightarrow H_{cr,eq}(X, \mathfrak{U}; V)$ we observe:

Proposition 2.4. *The image $j^n(f)$ of a continuous equivariant n -cocycle f on X in $\text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$ is cohomologous to the image $i_{eq}^n(f)$ of the equivariant n -cocycle $f \in A_{cr}^n(X, \mathfrak{U}; V)^G$ in $\text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$.*

Proof. The proof is a variation of the proof of [Fuc10, Proposition 14.3.8]: Let $f : X^{n+1} \rightarrow V$ be a continuous equivariant n -cocycle on X and (for $p + q = n - 1$) define equivariant cochains $\psi^{p,q} : X^{p+1} \times X^{q+1} \cong X^{n+1} \rightarrow V$ in $A_{cr}^{p,q}(X, \mathfrak{U}; V)^G$ via $\psi^{p,q}(\vec{x}, \vec{x}') = (-1)^p f(\vec{x}, \vec{x}')$. The vertical coboundary of the cochain $\psi^{p,q}$ is given by

$$\begin{aligned} [d_v \psi^{p,q}](\vec{x}, x'_0, \dots, x'_{q+1}) &= (-1)^p \sum (-1)^i f(\vec{x}, x'_0, \dots, \hat{x}'_i, \dots, x'_{q+1}) \\ &= - \sum (-1)^{p+1+i} f(x_0, \dots, \hat{x}_i, \dots, x_p, \vec{x}') \\ &= [d_h \psi^{p-1, q+1}](x_0, \dots, x_p, \vec{x}'). \end{aligned}$$

The anti-commutativity of the horizontal and the vertical differential ensures that the coboundary of the cochain $\sum_{p+q=n-1} (-1)^p \psi^{p,q}$ in the total complex is the cochain $j^n(f) - i_{eq}^n(f)$. Thus the cocycles $j^n(f)$ and $i_{eq}^n(f)$ are cohomologous in $\text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$. \square

Corollary 2.5. *The composition $H(i_{eq})^{-1}H(j_{eq}) : H_{c,eq}(X; V) \rightarrow H_{cr,eq}(X, \mathfrak{U}; V)$ is induced by the inclusion $A_c^*(X, \mathfrak{U}; V)^G \hookrightarrow A_{cr}^*(X, \mathfrak{U}; V)^G$.*

Corollary 2.6. *If the morphism $j_{eq}^* := j^{*G} : A_c^*(X; V)^G \rightarrow \text{Tot}A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cr}^*(X, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of X one can also consider the corresponding augmented colimit double complexes. In particular for the directed system of all open coverings of X one obtains the double complex

$$A_{cg}^{*,*}(X; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } X} A_{cr}^{*,*}(X; \mathfrak{U}; V)$$

whose rows and columns are augmented by the colimit complex $A_{cg}^*(X; V)$ and by the complex $A_c^*(X; V)$ respectively.

Lemma 2.7. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of X the morphism $\text{colim}_i i^* : \text{colim}_i A_{cr}^*(X, \mathfrak{U}_i; V) \rightarrow \text{Tot} \text{colim}_i A_{cr}^{*,*}(X, \mathfrak{U}_i; V)$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 2.1). \square

As a consequence the colimit morphism $i_{cg}^* : A_{cg}^*(X; V) \rightarrow \text{Tot}A_{cg}^{*,*}(X; V)$ induces an isomorphism in cohomology. The colimit double complex $A_{cg}^{*,*}(X; V)$ is a double complex of G -modules and the G -equivariant cochains in form a sub double complex $A_{cg}^{*,*}(X; V)^G$, whose rows and columns are augmented by the colimit complex $A_{cg,eq}^*(X; V)$ and by the complex $A_c^*(X; V)^G$ respectively.

Lemma 2.8. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of G -invariant open coverings of X the morphism $\text{colim}_i i_{eq}^* : \text{colim}_i A_{cr}^*(X, \mathfrak{U}_i; V)^G \rightarrow \text{Tot} \text{colim}_i A_{cr}^{*,*}(X, \mathfrak{U}_i; V)^G$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 2.3). \square

Moreover, since the open diagonal neighbourhoods $\mathfrak{U}[n]$ in X^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods, we observe:

Lemma 2.9. *The natural morphism of double complexes*

$$A_{cg,eq}^{*,*}(X; V) := \operatorname{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } X} A_{cr}^{*,*}(X; \mathfrak{U}; V)^G \rightarrow A_{cg}^*(X; V)^G$$

is a natural isomorphism.

Proof. The proof is analogous to that of Proposition 1.7. \square

As a consequence the colimit morphism $i_{cg,eq}^* : A_{cg,eq}^*(X; V) \rightarrow \operatorname{Tot} A_{cg}^{*,*}(X; V)^G$ induces an isomorphism in cohomology, and the morphism $H(i_{cg,eq})$ is invertible. For the composition $H(i_{cg,eq})^{-1} H(j_{eq}) : H_{c,eq}(X; V) \rightarrow H_{cg,eq}(X; \mathfrak{U}; V)$ we observe:

Proposition 2.10. *The image $j^n(f)$ of a continuous equivariant n -cocycle f on X in $\operatorname{Tot} A_{cg}^{*,*}(X; V)^G$ is cohomologous to the image $i_{cg,eq}^n(f)$ of the equivariant n -cocycle $f \in A_{cg,eq}^n(X; V)$ in $\operatorname{Tot} A_{cg}^{*,*}(X; V)^G$.*

Proof. The proof is analogous to that of Proposition 2.4. \square

Corollary 2.11. *The composition $H(i_{cg,eq})^{-1} H(j_{eq}) : H_{c,eq}(X; V) \rightarrow H_{cg,eq}(X; V)$ is induced by the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$.*

Corollary 2.12. *If the morphism $j_{eq}^* := j^* : A_c^*(X; V)^G \rightarrow \operatorname{Tot} A_{cg}^{*,*}(X; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg,eq}^*(X; V)$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

3. CONTINUOUS AND \mathfrak{U} -CONTINUOUS COCHAINS

In this section we consider transformation groups (G, X) and G -modules V for which we show that the inclusion $A_c^*(X; \mathfrak{U}; V)^G \hookrightarrow A_{cr}^*(X; \mathfrak{U}; V)^G$ of the complex of continuous equivariant cochains into the complex of equivariant \mathfrak{U} -continuous cochains induces an isomorphism $H_c^*(X; \mathfrak{U}; V) \cong H_{cr}^*(X; \mathfrak{U}; V)$ provided the topological space X is contractible. The proof relies on the row exactness of the double complexes $A_c^{*,*}(X; \mathfrak{U}; V)^G$ and $A_{cr}^{*,*}(X; \mathfrak{U}; V)^G$. At first we reduce the problem to the non-equivariant case:

Proposition 3.1. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cr}^{p,*}(X; \mathfrak{U}; V)$ are exact, then the augmented sub column complexes $A_c^p(X; V)^G \hookrightarrow A_{cr}^{p,*}(X; \mathfrak{U}; V)^G$ of equivariant cochains are exact as well.*

Proof. Assume that the augmented column complexes $A_c^p(X; \mathfrak{U}; V) \hookrightarrow A_{cr}^{p,*}(X; \mathfrak{U}; V)$ are exact. Then each equivariant vertical cocycle $f_{eq}^{p,q} \in A_{rc}^{p,q}(X; \mathfrak{U}; V)^G$ is the vertical coboundary $d_v f^{p,q-1}$ of a cocycle $f^{p,q-1} \in A_{cr}^{p,q-1}(X; \mathfrak{U}; V)$ (which is not necessary equivariant). Define an equivariant cochain $f_{eq}^{p,q-1}$ of bidegree $(p, q-1)$ via

$$f_{eq}^{p,q-1}(\vec{x}, \vec{x}') := x_0 \cdot f^{p,q-1}(x_0^{-1} \cdot \vec{x}, x_0^{-1} \cdot \vec{x}').$$

This equivariant cochain is continuous on $X^{p+1} \times \mathfrak{U}[q-1]$ because $f^{p,q-1}$ is continuous on $X^{p+1} \times \mathfrak{U}[q-1]$. We assert that the vertical coboundary $d_v f_{eq}^{p,q-1}$ of f_{eq} is

the equivariant vertical cocycle $f_{eq}^{p,q}$. Indeed, since the differential d_v is equivariant, the vertical coboundary of $f_{eq}^{p,q-1}$ computes to

$$d_v f_{eq}^{p,q-1}(\vec{x}, \vec{x}') = x_0 \cdot [d_v f_{eq}^{p,q-1}(x_0^{-1} \cdot \vec{x}, x_0^{-1} \cdot \vec{x}')] = f_{eq}^{p,q}(\vec{x}, \vec{x}').$$

Thus every equivariant vertical cocycle $f_{eq}^{p,q}$ in $A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$ is the vertical coboundary of an equivariant cochain $f_{eq}^{p,q-1}$ of bidegree $(p, q-1)$. \square

Corollary 3.2. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cr}^{p,*}(X, \mathfrak{U}; V)$ are exact, then the inclusion $j_{eq}^* : A_c^*(X; V)^G \hookrightarrow \text{Tot} A_{cr}^{*,*}(X, \mathfrak{U}, V)^G$ induces an isomorphism in cohomology.*

Corollary 3.3. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cr}^{p,*}(X, \mathfrak{U}; V)$ are exact, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cr}^*(X, \mathfrak{U}; V)^G$ induces an isomorphism in cohomology.*

To achieve the announced result it remains to show that for contractible spaces X the colimit augmented columns $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V)$ are exact. For this purpose we first consider the cochain complex associated to the cosimplicial abelian group $A^{p,*}(X; V) := \{f : X^{p+1} \times X^{*+1} \rightarrow V \mid \forall \vec{x}' \in X^{*+1} : f(-, \vec{x}') \in C(X^{p+1}, V)\}$ of global cochains, its subcomplex $A_{cr}^{p,*}(X, \mathfrak{U}; V)$ and the cochain complexes associated to the cosimplicial abelian groups

$$\begin{aligned} A^{p,*}(\mathfrak{U}; V) &:= \{f : X^{p+1} \times \mathfrak{U}[*] \rightarrow V \mid \forall \vec{x}' \in \mathfrak{U}[*] : f(-, \vec{x}') \in C(X^{p+1}, V)\} \quad \text{and} \\ A_c^{p,*}(X, \mathfrak{U}; V) &:= C(X^{p+1} \times \mathfrak{U}[*], V). \end{aligned}$$

Restriction of global to local cochains induces morphisms of cochain complexes $\text{Res}^{p,*} : A^{p,*}(X; V) \rightarrow A^{p,*}(X, \mathfrak{U}; V)$ and $\text{Res}_{cr}^{p,*} : A_{cr}^{p,*}(X, \mathfrak{U}; V) \rightarrow A_c^{p,*}(X, \mathfrak{U}; V)$ intertwining the inclusions of the subcomplexes $A_{cr}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A^{p,*}(X; V)$ and $A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A^{p,*}(X, \mathfrak{U}; V)$, so one obtains the following commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker(\text{Res}_{cr}^{p,*}) & \longrightarrow & A_{cr}^{p,*}(X, \mathfrak{U}; V) & \longrightarrow & A_c^{p,*}(X, \mathfrak{U}; V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(\text{Res}^{p,*}) & \longrightarrow & A^{p,*}(X; V) & \longrightarrow & A^{p,*}(X, \mathfrak{U}; V) \longrightarrow 0 \end{array}$$

of cochain complexes whose rows are exact. The kernel $\ker(\text{Res}^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $X^{p+1} \times \mathfrak{U}[q]$. Since these (p, q) -cochains are continuous on $X^{p+1} \times \mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}^{p,*}) = \ker(\text{Res}_{cr}^{p,*})$ by $K^{p,*}$ and denote the cohomology groups of the complex $A_{cr}^{p,*}(X, \mathfrak{U}; V)$ by $H_{cr}^{p,*}(X, \mathfrak{U}; V)$, the cohomology groups of the complex $A_c^{p,*}(X, \mathfrak{U}; V)$ of continuous cochains by $H_c^{p,*}(X, \mathfrak{U}; V)$ and the cohomology groups of the complex $A^{p,*}(X, \mathfrak{U}; V)$ by $H^{p,*}(X, \mathfrak{U}; V)$.

Lemma 3.4. *The cochain complexes $A^{p,*}(X; V)$ are exact.*

Proof. For any point $* \in X$ the homomorphisms $h^{p,q} : A^{p,q}(X; V) \rightarrow A^{p,q-1}(X; V)$ given by $h^{p,q}(f)(\vec{x}, \vec{x}') := f(\vec{x}, *, \vec{x}')$ form a contraction of the complex $A^{p,*}(X; V)$. \square

The morphism of short exact sequences of cochain complexes in Diagram 3.1 gives rise to a morphism of long exact cohomology sequences, in which the cohomology

of the complex $A^{p,*}(X; V)$ is trivial:

$$(3.2) \quad \begin{array}{ccccccc} \longrightarrow & H^q(K^{p,*}) & \longrightarrow & H_{cr}^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H_c^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H^{q+1}(K^{p,*}) & \longrightarrow \\ & \parallel & & \downarrow & & \downarrow & & \parallel & \\ \xrightarrow{\cong} & H^q(K^{p,*}) & \longrightarrow & 0 & \longrightarrow & H^{p,q}(X, \mathfrak{U}; V) & \xrightarrow{\cong} & H^{q+1}(K^{p,*}) & \longrightarrow \end{array}$$

Lemma 3.5. *The augmented complex $A_c^p(X; V) \hookrightarrow A_{cr}^{p,*}(X, \mathfrak{U}; V)$ is exact if and only if the inclusion $A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A^{p,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology.*

Proof. This is an immediate consequence of Diagram 3.2 □

Proposition 3.6. *If the inclusion $A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A^{p,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology, then the inclusions $j_{eq}^* : A_c^*(X; V)^G \hookrightarrow \text{Tot} A_{cr}^{*,*}(X, \mathfrak{U}; V)^G$ and $A_c^*(X, \mathfrak{U}; V)^G \hookrightarrow A_{cr}^*(X, \mathfrak{U}; V)^G$ also induces an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 3.2 and 3.3. □

The passage to the colimit over all open coverings of X yields the corresponding results for the complexes of cochains with continuous germs:

Proposition 3.7. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V)$ are exact, then the augmented sub column complexes $A_c^p(X; V)^G \hookrightarrow A_{cg}^{p,*}(X; V)^G$ of equivariant cochains are exact as well.*

Proof. The proof is similar to that of Proposition 3.1. □

Corollary 3.8. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V)$ are exact, then the inclusion $j_{eq}^* : A_c^*(X; V)^G \hookrightarrow \text{Tot} A_{cg}^{*,*}(X; V)^G$ induces an isomorphism in cohomology.*

Corollary 3.9. *If the augmented column complexes $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V)$ are exact, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ induces an isomorphism in cohomology.*

Remark 3.10. Alternatively to taking the colimit over all open coverings \mathfrak{U} of X one may consider G -invariant open coverings only to obtains the same results. (This was shown in Proposition 1.7 and Lemmata 2.9.)

Example 3.11. If $G = X$ is a topological group which acts on itself by left translation and the augmented columns $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V) := \text{colim} A^{p,*}(X, \mathfrak{U}_U; V)$ (where U runs over all open identity neighbourhoods in G) are exact, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ induces an isomorphism in cohomology.

The complex $A^{p,*}(X, \mathfrak{U}; V)$ is isomorphic to the complex $A^*(\mathfrak{U}; C(X^{p+1}, V))$. The colimit $A_{AS}^*(X; C(X^{p+1}, V)) := \text{colim} A^*(\mathfrak{U}; C(X^{p+1}, V))$, where \mathfrak{U} runs over all open coverings of X is the complex of Alexander-Spanier cochains on X . Therefore the colimit complex $\text{colim} A^p(X; A^*(\mathfrak{U}; V))$ is isomorphic to the cochain complex $A_{AS}^*(X; C(X^{p+1}, V))$. A similar observation can be made for the cochain complex $A_c^{p,*}(X, \mathfrak{U}; V)$ if the exponential law $C(X^{p+1} \times \mathfrak{U}[q], V) \cong C(X, C(\mathfrak{U}[q], V))$ holds for a cofinal set of open coverings \mathfrak{U} of X . Passing to the colimit in Diagram 3.1 yields the morphism

$$\begin{array}{ccccccc}
(3.3) & 0 & \longrightarrow & \ker(\text{Res}_{cg}^{p,*}) & \longrightarrow & A_c^{p,*}(X; V) & \longrightarrow & \text{colim } A_c^{p,*}(X, \mathfrak{U}; V) & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
& 0 & \longrightarrow & \ker(\text{Res}^{p,*}) & \longrightarrow & A^{p,*}(X; V) & \longrightarrow & A_{AS}^*(X; C^{p+1}(X, V)) & \longrightarrow & 0
\end{array}$$

of short exact sequences of cochain complexes. The kernel $\ker(\text{Res}^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $X^{p+1} \times \mathfrak{U}[q]$ for some open covering \mathfrak{U} of X . Since these (p, q) -cochains are continuous on $X^{p+1} \times \mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}^{p,*}) = \ker(\text{Res}_{cg}^{p,*})$ by $K_{cg}^{p,*}$ and denote the cohomology groups of the complex $A_c^{p,*}(X; V)$ by $H_{cg}^{p,*}(X; V)$. The morphism of short exact sequences of cochain complexes in Diagram 3.3 gives rise to a morphism of long exact cohomology sequences:

$$\begin{array}{ccccccc}
(3.4) & \longrightarrow & H^q(K_{cg}^{p,*}) & \longrightarrow & H_{cg}^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H^q(\text{colim } A_c^{p,*}(X, \mathfrak{U}; V)) & \longrightarrow & H^{q+1}(K_{cg}^{p,*}) & \longrightarrow \\
& & \parallel & & \downarrow & & \downarrow & & \parallel & \\
& \xrightarrow{\cong} & H^q(K^{p,*}) & \longrightarrow & 0 & \longrightarrow & H_{AS}^q(X; C^{p+1}(X, V)) & \xrightarrow{\cong} & H^{q+1}(K^{p,*}) & \longrightarrow
\end{array}$$

Lemma 3.12. *The augmented complex $A_c^p(X; V) \hookrightarrow A_{cg}^{p,*}(X; V)$ is exact if and only if the inclusion $\text{colim } A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C^{p+1}(X, V))$ of cochain complexes induces an isomorphism in cohomology.*

Proof. This is an immediate consequence of Diagram 3.4 □

Proposition 3.13. *If the inclusion $\text{colim } A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C(X^{p+1}, V))$ induces an isomorphism in cohomology, then $j_{eq}^* : A_c^*(X; V)^G \hookrightarrow \text{Tot } A_{cg}^*(X; V)^G$ and $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ also induce an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 3.8 and 3.9. □

As observed before (cf. Remark 3.10) one may restrict oneself to the directed system of G -invariant open coverings only to achieve the same result. Thus we observe:

Corollary 3.14. *If $G = X$ is a locally contractible topological group which acts on itself by left translation and the inclusion $\text{colim } A_c^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C(X^{p+1}, V))$ (where \mathfrak{U} runs over all open identity neighbourhoods in G) induces an isomorphism in cohomology, then the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ induces an isomorphism in cohomology as well.*

Proof. It has been shown in [vE62] that the cohomology of the colimit cochain complex $\text{colim } A^*(\mathfrak{U}; C(X^{p+1}, V))$ is the Alexander-Spanier cohomology of X . □

Lemma 3.15. *If the topological space X is contractible, then the cohomology of the complex $\text{colim } A_c^{p,*}(X, \mathfrak{U}; V)$ is trivial.*

Proof. The reasoning is analogous to that for the Alexander-Spanier presheaf. The proof [Fuc10, Theorem 2.5.2] carries over almost in verbatim. □

Theorem 3.16. *For contractible X the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ induces an isomorphism in cohomology.*

Proof. If the topological space X is contractible, then the Alexander-Spanier cohomology $H_{AS}(X; C^{p+1}(X, V))$ is trivial and the cohomology of the cochain complex $\text{colim } A_c^{p,*}(X, \mathfrak{U}; V)$ is trivial by Lemma 3.15. By Proposition 3.13 the inclusion $A_c^*(X; V)^G \hookrightarrow A_{cg}^*(X; V)^G$ then induces an isomorphism in cohomology. \square

Corollary 3.17. *For contractible topological groups G the continuous group cohomology is isomorphic to the cohomology of homogeneous group cochains with continuous germ at the diagonal.*

4. WORKING IN THE CATEGORY OF k -SPACES

In this section we consider transformation groups (G, X) in the category \mathbf{kTop} of k -spaces and G -modules V in \mathbf{kTop} . Working only in the category \mathbf{kTop} we construct a spectral sequence analogously to that in Section 2 and derive results analogous to those obtained there.

Definition 4.1. For every k -space X and abelian k -group V the subcomplex $A_{kc}^*(X; V) := C(kX^{*+1}; V)$ of the standard complex is called the *continuous standard complex in \mathbf{kTop}* .

For open coverings \mathfrak{U} of a k -space X we also consider the subcomplex of $A^*(X; V)$ formed by the groups

$$A_{kcr}^n(X, \mathfrak{U}; V) := \{f \in A^n(X; V) \mid f|_{k\mathfrak{U}[n]} \in C(k\mathfrak{U}[n]; V)\}$$

of cochains whose restriction to the open subspaces $k\mathfrak{U}[n]$ of kX^{n+1} are continuous. The cohomology of the cochain complex $A_{kcr}^*(X, \mathfrak{U}; V)$ is denoted by $H_{kcr}(X, \mathfrak{U}; V)$. If the covering \mathfrak{U} of X is G -invariant, then the subspaces $k\mathfrak{U}[*]$ is a simplicial G -subspace of the simplicial G -space kX^{*+1} .

Example 4.2. If $G = X$ is a k -group which acts on itself by left translation and U an open identity neighbourhood, then $\mathfrak{U}_U := \{g.U \mid g \in G\}$ is a G -invariant open covering of G and $k\mathfrak{U}[*]$ is a simplicial G -subspace of kG^{*+1} .

For G -invariant coverings \mathfrak{U} of X the cohomology of the subcomplex $A_{kcr}^*(X, \mathfrak{U}; V)^G$ of G -equivariant cochains is denoted by $H_{kcr,eq}(X, \mathfrak{U}; V)$.

Example 4.3. If $G = X$ is a k -group which acts on itself by left translation and U an open identity neighbourhood, then the complex $A_{kcr}^*(X, \mathfrak{U}_U; V)^G$ is the complex of homogeneous group cochains whose restrictions to the subspaces $k\mathfrak{U}_U[*]$ are continuous. (These are sometimes called \mathfrak{U} -continuous cochains.)

For directed systems $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of X one can also consider the colimit complex $\text{colim}_i A_{kcr}^*(X, \mathfrak{U}_i; V)$. In particular, if the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods, one obtains the complex

$$A_{kcg}^*(X; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } X} A_{kcr}^*(X; \mathfrak{U}; V)$$

of global cochains whose germs at the diagonal are continuous. This happens for all k -spaces X for which the finite products X^{n+1} in \mathbf{Top} are already compactly Hausdorff generated, e.g. metrisable spaces, locally compact spaces or Hausdorff k_ω -spaces. The complex $A_{kcg}^*(X; V)$ is then a subcomplex of the standard complex $A^*(X; V)$ which is invariant under the G -action (Eq. 1.1) and thus a sub complex of G -modules. The G -equivariant cochains with continuous germ form a subcomplex

$A_{kcg}^*(X; V)^G$ thereof, whose cohomology is denoted by $H_{kcg,eq}(X; V)$. The latter subcomplex can also be obtained by taking the colimit over all G -invariant open coverings of X only:

Proposition 4.4. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the natural morphism of cochain complexes*

$$A_{kcg,eq}^*(X; V) := \operatorname{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } X} A_{kcr}^*(X; \mathfrak{U}; V)^G \rightarrow A_{kcg}^*(X; V)^G$$

is a natural isomorphism.

Proof. The proof is analogous to that of Proposition 1.7. \square

Corollary 4.5. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the cohomology $H_{kcg,eq}(X; V)$ is the cohomology of the complex of equivariant cochains which are continuous on some G -invariant neighbourhood of the diagonal.*

Example 4.6. If $G = X$ is a metrisable or locally compact topological group or a real or complex Kac-Moody group which acts on itself by left translation, then the complex $A_{kcg}^*(G; V)^G$ is the complex of homogeneous group cochains whose germs at the diagonal are continuous. (By abuse of language these are sometimes called 'locally continuous' group cochains.)

Analogously to the procedure in Section 2 we can construct a spectral sequence relating $A_{kcr}^*(X, \mathfrak{U}; V)$ and $A_{kc}^*(X; V)$. For this purpose we consider the abelian groups

$$(4.1) \quad A_{kcr}^{p,q}(X, \mathfrak{U}; V) := \{ f : X^{p+1} \times X^{q+1} \rightarrow V \mid f|_{[kX^{p+1} \times_k k\mathfrak{U}[q]} \text{ is continuous} \}.$$

The abelian groups $A_{kcr}^{p,q}(X, \mathfrak{U}; V)$ form a first quadrant double complex whose vertical and horizontal differentials are given by the same formulas as for the double complex $A_{cr}^{p,q}(X, \mathfrak{U}; V)$ introduced in Section 2. Analogously to the latter double complex the rows of the double complex $A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ can be augmented by the complex $A_{kcr}^*(X, \mathfrak{U}; V)$ for the covering \mathfrak{U} and the columns can be augmented by the exact complex $A_{kc}^*(X; V)$ of continuous cochains. We denote the total complex of the double complex $A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ by $\operatorname{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)$. The augmentations of the rows and columns induce morphisms $i_k^* : A_{kcr}^*(X, \mathfrak{U}; V) \rightarrow \operatorname{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ and $j_k^* : A_{kc}^*(X; V) \rightarrow \operatorname{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ of cochain complexes respectively.

Lemma 4.7. *The morphism $i_k^* : A_{kcr}^*(X, \mathfrak{U}; V) \rightarrow \operatorname{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology.*

Proof. The proof of Lemma 2.1 also works in the category **kTop** of k -spaces. \square

For G -invariant open coverings \mathfrak{U} of X one can consider the sub double complex $A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$ of $A_{kcr}^{*,*}(X, \mathfrak{U}; V)$ whose rows are augmented by the cochain complex $A_{kcr}^*(X, \mathfrak{U}; V)^G$ for the covering \mathfrak{U} and the columns can be augmented by the complex $A_{kc}^*(X; V)^G$ of continuous equivariant cochains (which is not exact in general).

Lemma 4.8. *For G -invariant coverings \mathfrak{U} of X the morphism $i_{k,eq}^* := i_k^{*G}$ induces an isomorphism in cohomology.*

Proof. The proof is analogous to that of Lemma 2.3. \square

So the morphism $H(i_{k,eq}) : H_{kcr,eq}(X, \mathfrak{U}; V) \rightarrow H(\text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G)$ is invertible. For the composition $H(i_{k,eq})^{-1} H(j_{k,eq}) : H_{kc,eq}(X; V) \rightarrow H_{kcr,eq}(X, \mathfrak{U}; V)$ we observe:

Proposition 4.9. *The image $j_k^n(f)$ of a continuous equivariant n -cocycle f on X in $\text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$ is cohomologous to the image $i_{k,eq}^n(f)$ of the equivariant n -cocycle $f \in A_{kcr}^n(X, \mathfrak{U}; V)^G$ in $\text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$.*

Proof. The proof is analogous to that of Proposition 2.4. \square

Corollary 4.10. *The map $H(i_{k,eq})^{-1} H(j_{eq}) : H_{kc,eq}(X; V) \rightarrow H_{kcr,eq}(X, \mathfrak{U}; V)$ is induced by the inclusion $A_{kc}^*(X, \mathfrak{U}; V)^G \hookrightarrow A_{kcr}^*(X, \mathfrak{U}; V)^G$.*

Corollary 4.11. *If the morphism $j_{k,eq}^* := j^{*G} : A_{kc}^*(X; V)^G \rightarrow \text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcr}^*(X, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

Lemma 4.12. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of X the morphism $\text{colim}_i i^* : \text{colim}_i A_{kcr}^*(X, \mathfrak{U}_i; V) \rightarrow \text{Tot} \text{colim}_i A_{kcr}^{*,*}(X, \mathfrak{U}_i; V)$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 4.7). \square

Lemma 4.13. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of G -invariant open coverings of X the morphism $\text{colim}_i i_{k,eq}^* : \text{colim}_i A_{kcr}^*(X, \mathfrak{U}_i; V)^G \rightarrow \text{Tot} \text{colim}_i A_{kcr}^{*,*}(X, \mathfrak{U}_i; V)^G$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 4.8). \square

If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then one obtains the double complex complex

$$A_{kcg}^{*,*}(X; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } X} A_{kcr}^{*,*}(X; \mathfrak{U}; V)$$

whose rows and columns are augmented by the complexes $A_{kcg}^*(X; V)$ and $A_{kc}^*(X; V)$ respectively. In this case the colimit morphism $i_{kcg}^* : A_{kcg}^*(X; V) \rightarrow \text{Tot} A_{kcg}^{*,*}(X; V)$ induces an isomorphism in cohomology. Furthermore the colimit double complex $A_{kcg}^{*,*}(X; V)$ then is a double complex of G -modules and the G -equivariant cochains in form a sub double complex $A_{kcg}^{*,*}(X; V)^G$, whose rows and columns are augmented by the colimit complex $A_{kcg,eq}^*(X; V)$ and by the complex $A_{kc}^*(X; V)^G$ respectively. In addition we observe:

Lemma 4.14. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the natural morphism of double complexes*

$$A_{kcg,eq}^{*,*}(X; V) := \text{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } X} A_{kcr}^{*,*}(X; \mathfrak{U}; V)^G \rightarrow A_{kcg}^*(X; V)^G$$

is a natural isomorphism.

Proof. The proof is analogous to that of Proposition 4.4. \square

As a consequence the colimit morphism $i_{kcg,eq}^* : A_{kcg,eq}^*(X; V) \rightarrow \text{Tot} A_{kcg}^{*,*}(X; V)^G$ then induces an isomorphism in cohomology, and the morphism $H(i_{kcg,eq})$ is invertible. For the composition $H(i_{kcg,eq})^{-1} H(j_{k,eq}) : H_{kcg,eq}(X; V) \rightarrow H_{kcg,eq}(X, \mathfrak{U}; V)$ we observe:

Proposition 4.15. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the image $j^n(f)$ of a continuous equivariant n -cocycle f on X in $\text{Tot} A_{kcg}^{*,*}(X; , V)^G$ is cohomologous to the image $i_{kcg,eq}^n(f)$ of the equivariant cocycle $f \in A_{kcg,eq}^n(X; V)$ in $\text{Tot} A_{kcg}^{*,*}(X; V)^G$.*

Proof. The proof is analogous to that of Proposition 2.4. \square

Corollary 4.16. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the composition $H(i_{kcg,eq})^{-1} H(j_{k,eq}) : H_{kcg,eq}(X; V) \rightarrow H_{kcg,eq}(X; V)$ is induced by the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$.*

Corollary 4.17. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods and the morphism $j_{k,eq}^* := j_k^{*G} : A_{kc}^*(X; V)^G \rightarrow \text{Tot} A_{kcg}^{*,*}(X; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg,eq}^*(X; V)$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

5. CONTINUOUS AND \mathfrak{U} -CONTINUOUS COCHAINS ON k -SPACES

In this section we consider transformation k -groups (G, X) and G -modules V in \mathbf{kTop} for which we show that the inclusion $A_{kc}^*(X, \mathfrak{U}; V)^G \hookrightarrow A_{kcr}^*(X, \mathfrak{U}; V)^G$ of the complex of continuous equivariant cochains into the complex of equivariant \mathfrak{U} -continuous cochains induces an isomorphism $H_{kc}^*(X, \mathfrak{U}; V) \cong H_{kcr}^*(X, \mathfrak{U}; V)$ provided the k -space X is contractible. The proceeding is similar to that in Section 5. At first we reduce the problem to the non-equivariant case:

Proposition 5.1. *If the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcr}^{p,*}(X, \mathfrak{U}; V)$ are exact, then the augmented sub column complexes $A_{kc}^p(X; V)^G \hookrightarrow A_{kcr}^{p,*}(X, \mathfrak{U}; V)^G$ of equivariant cochains are exact as well.*

Proof. The proof is analogous to that of Proposition 3.1. \square

Corollary 5.2. *If the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcr}^{p,*}(X, \mathfrak{U}; V)$ are exact, then the inclusion $j_{k,eq}^* : A_{kc}^*(X; V)^G \hookrightarrow \text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$ induces an isomorphism in cohomology.*

Corollary 5.3. *If the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcr}^{p,*}(X, \mathfrak{U}; V)$ are exact, then the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcr}^*(X, \mathfrak{U}; V)^G$ induces an isomorphism in cohomology.*

To achieve the announced result it remains to show that for contractible k -spaces X the colimit augmented columns $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V)$ are exact. For this purpose we first consider the cochain complex associated to the cosimplicial abelian group

$$A_k^{p,*}(X; V) := \{f : X^{p+1} \times X^{*+1} \rightarrow V \mid \forall \vec{x}' \in X^{*+1} : f(-, \vec{x}') \in C(kX^{p+1}, V)\}$$

of global cochains, its subcomplex $A_{kcr}^{p,*}(X, \mathfrak{U}; V)$ and the cochain complexes associated to the cosimplicial abelian groups

$$\begin{aligned} A_k^{p,*}(\mathfrak{U}; V) &:= \{f : X^{p+1} \times \mathfrak{U}[*] \rightarrow \mid \forall \vec{x}' \in \mathfrak{U}[*] : f(-, \vec{x}') \in C(kX^{p+1}, V)\} \quad \text{and} \\ A_{kc}^{p,*}(X, \mathfrak{U}; V) &:= C(kX^{p+1} \times_k k\mathfrak{U}[*], V). \end{aligned}$$

Restriction of global to local cochains induces morphisms of cochain complexes $\text{Res}_k^{p,*} : A_k^{p,*}(X; V) \rightarrow A_k^{p,*}(X, \mathfrak{U}; V)$ and $\text{Res}_{kcr}^{p,*} : A_{kcr}^{p,*}(X, \mathfrak{U}; V) \rightarrow A_{kc}^{p,*}(X, \mathfrak{U}; V)$ intertwining the inclusions of the subcomplexes $A_{kcr}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_k^{p,*}(X; V)$ and $A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_k^{p,*}(X, \mathfrak{U}; V)$, so one obtains the following commutative diagram

$$(5.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker(\text{Res}_{kcr}^{p,*}) & \longrightarrow & A_{kcr}^{p,*}(X, \mathfrak{U}; V) & \longrightarrow & A_{kc}^{p,*}(X, \mathfrak{U}; V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(\text{Res}_k^{p,*}) & \longrightarrow & A_k^{p,*}(X; V) & \longrightarrow & A_k^{p,*}(X, \mathfrak{U}; V) \longrightarrow 0 \end{array}$$

of cochain complexes whose rows are exact. The kernel $\ker(\text{Res}_k^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $kX^{p+1} \times_k k\mathfrak{U}[q]$. Since these (p, q) -cochains are continuous on $kX^{p+1} \times_k k\mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}_k^{p,*}) = \ker(\text{Res}_{kcr}^{p,*})$ by $K_k^{p,*}$ and denote the cohomology groups of the complex $A_{kcr}^{p,*}(X, \mathfrak{U}; V)$ by $H_{kcr}^{p,*}(X, \mathfrak{U}; V)$, the cohomology groups of the complex $A_{kc}^{p,*}(X, \mathfrak{U}; V)$ of continuous cochains by $H_{kc}^{p,*}(X, \mathfrak{U}; V)$ and the cohomology groups of the complex $A_k^{p,*}(X, \mathfrak{U}; V)$ by $H_k^{p,*}(X, \mathfrak{U}; V)$.

Lemma 5.4. *The cochain complexes $A_k^{p,*}(X; V)$ are exact.*

Proof. For any point $* \in X$ the homomorphisms $h^{p,q} : A_k^{p,q}(X; V) \rightarrow A_k^{p,q-1}(X; V)$ given by $h^{p,q}(f)(\vec{x}, \vec{x}') := f(\vec{x}, *, \vec{x}')$ form a contraction of the complex $A_k^{p,*}(X; V)$. \square

The morphism of short exact sequences of cochain complexes in Diagram 5.1 gives rise to a morphism of long exact cohomology sequences, in which the cohomology of the complex $A_k^{p,*}(X; V)$ is trivial:

$$(5.2) \quad \begin{array}{ccccccc} \longrightarrow & H^q(K_k^{p,*}) & \longrightarrow & H_{kcr}^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H_{kc}^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H^{q+1}(K_k^{p,*}) \longrightarrow \\ & \parallel & & \downarrow & & \downarrow & & \parallel \\ \xrightarrow{\cong} & H^q(K^{p,*}) & \longrightarrow & 0 & \longrightarrow & H_k^{p,q}(X, \mathfrak{U}; V) & \xrightarrow{\cong} & H^{q+1}(K^{p,*}) \longrightarrow \end{array}$$

Lemma 5.5. *The augmented complex $A_{kc}^p(X; V) \hookrightarrow A_{cr}^{p,*}(X, \mathfrak{U}; V)$ is exact if and only if the inclusion $A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_k^{p,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology.*

Proof. This is an immediate consequence of Diagram 5.2 \square

Proposition 5.6. *If the inclusion $A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_k^{p,*}(X, \mathfrak{U}; V)$ induces an isomorphism in cohomology, then the inclusions $A_c^*(X; V)^G \hookrightarrow \text{Tot} A_{kcr}^{*,*}(X, \mathfrak{U}; V)^G$ and $A_{kc}^*(X, \mathfrak{U}; V)^G \hookrightarrow A_{kcr}^{p,*}(X, \mathfrak{U}; V)^G$ also induces an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 5.2 and 5.3. \square

For k -spaces X for which the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods the passage to the colimit over all open coverings of X yields the corresponding results for the complexes of cochains with continuous germs:

Proposition 5.7. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods and the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V)$ are exact, then the augmented sub column complexes $A_{kc}^p(X; V)^G \hookrightarrow A_{kcg}^{p,*}(X; V)^G$ of equivariant cochains are exact as well.*

Proof. The proof is similar to that of Proposition 3.1. \square

Corollary 5.8. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods and the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V)$ are exact, then the inclusion $j_{k,eq}^* : A_{kc}^*(X; V)^G \hookrightarrow \text{Tot} A_{kcg}^{*,*}(X; V)^G$ induces an isomorphism in cohomology.*

Corollary 5.9. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods and the augmented column complexes $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V)$ are exact, then the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ induces an isomorphism in cohomology.*

Remark 5.10. Alternatively to taking the colimit over all open coverings \mathfrak{U} of X one may consider G -invariant open coverings only to obtain the same results. (This was shown in Proposition 4.4 and Lemmata 4.14.)

Example 5.11. If $G = X$ is a metrisable, locally compact or Hausdorff k_ω topological group which acts on itself by left translation and the augmented columns $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V) := \text{colim} A_k^{p,*}(X, \mathfrak{U}_U; V)$ (where U runs over all open identity neighbourhoods in G) are exact, then $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ induces an isomorphism in cohomology.

The complex $A^{p,*}(X, \mathfrak{U}; V)$ is isomorphic to the complex $A^*(\mathfrak{U}; C(kX^{p+1}, V))$. If the open diagonal neighbourhoods $\mathfrak{U}[n]$ in X^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods then the colimit $A_{AS}^*(X; C(X^{p+1}, V)) := \text{colim} A^*(\mathfrak{U}; C(X^{p+1}, V))$, where \mathfrak{U} runs over all open coverings of X is the complex of Alexander-Spanier cochains on X (with values in $C(X^{p+1}, V)$). In this case the colimit complex $\text{colim} A^p(X; A^*(\mathfrak{U}; V))$ is isomorphic to the cochain complex $A_{AS}^*(X; C(kX^{p+1}, V))$. A similar observation can be made for the cochain complex $A_{kc}^{p,*}(X, \mathfrak{U}; V)$ because the exponential law $C(kX^{p+1} \times_k k\mathfrak{U}[q], V) \cong C(X, kC(\mathfrak{U}[q], V))$ holds in \mathbf{kTop} . Passing to the colimit in Diagram 5.1 yields the morphism

$$(5.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker(\text{Res}_{kcg}^{p,*}) & \longrightarrow & A_{kcg}^{p,*}(X; V) & \longrightarrow & \text{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(\text{Res}_k^{p,*}) & \longrightarrow & A_k^{p,*}(X; V) & \longrightarrow & A_{AS}^*(X; C^{p+1}(X, V)) \longrightarrow 0 \end{array}$$

of short exact sequences of cochain complexes. The kernel $\ker(\text{Res}_k^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $kX^{p+1} \times_k k\mathfrak{U}[q]$ for some open covering \mathfrak{U} of X . Since these (p, q) -cochains are continuous on $kX^{p+1} \times_k k\mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}_k^{p,*}) = \ker(\text{Res}_{kcg}^{p,*})$ by $K_{kcg}^{p,*}$ and denote the cohomology groups of the complex $A_{kcg}^{p,*}(X; V)$ by $H_{kcg}^{p,*}(X; V)$. The morphism of short exact sequences of cochain complexes in Diagram 5.3 gives

rise to a morphism of long exact cohomology sequences:

$$(5.4) \quad \begin{array}{ccccccc} \longrightarrow & H^q(K_{kcg}^{p,*}) & \longrightarrow & H_{kcg}^{p,q}(X, \mathfrak{U}; V) & \longrightarrow & H^q(\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V) & \longrightarrow & H^{q+1}(K_{kcg}^{p,*}) & \longrightarrow \\ & \parallel & & \downarrow & & \downarrow & & \parallel & \\ \longrightarrow & \cong H^q(K_{kcg}^{p,*}) & \longrightarrow & 0 & \longrightarrow & H_{AS}^q(X; C^{p+1}(X, V)) & \xrightarrow{\cong} & H^{q+1}(K_{kcg}^{p,*}) & \longrightarrow \end{array}$$

Lemma 5.12. *The augmented complex $A_{kc}^p(X; V) \hookrightarrow A_{kcg}^{p,*}(X; V)$ is exact if and only if the inclusion $\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C^{p+1}(X, V))$ of cochain complexes induces an isomorphism in cohomology.*

Proof. This is an immediate consequence of Diagram 5.4 □

Proposition 5.13. *If the open diagonal neighbourhoods $k\mathfrak{U}[n]$ in kX^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods and the inclusion $\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C(X^{p+1}, V))$ induces an isomorphism in cohomology, then $j_{k,eq}^* : A_{kc}^*(X; V)^G \hookrightarrow \operatorname{Tot} A_{kcg}^{*,*}(X; V)^G$ and $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ also induce an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 5.8 and 5.9. □

As observed before (cf. Remark 5.10) one may restrict oneself to the directed system of G -invariant open coverings only to achieve the same result. Thus we observe:

Corollary 5.14. *If $G = X$ is a locally contractible metrisable, locally contractible locally compact or locally contractible Hausdorff k_ω topological group which acts on itself by left translation and the inclusion $\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(X; C(X^{p+1}, V))$ (where \mathfrak{U} runs over all open identity neighbourhoods in G) induces an isomorphism in cohomology, then the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ induces an isomorphism in cohomology as well.*

Proof. It has been shown in [vE62] that the cohomology of the colimit cochain complex $\operatorname{colim} A^*(\mathfrak{U}; C(kX^{p+1}, V))$ is the Alexander-Spanier cohomology of X with coefficients $C(kX^{p+1}, V)$. □

Lemma 5.15. *If the topological space X is contractible, then the cohomology of the complex $\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V)$ is trivial.*

Proof. The reasoning is analogous to that for the Alexander-Spanier presheaf. The proof [Fuc10, Theorem 2.5.2] carries over almost in verbatim. □

Theorem 5.16. *For contractible X the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ induces an isomorphism in cohomology.*

Proof. If the k -space X is contractible, then the Alexander-Spanier cohomology of X is trivial and the cohomology of the cochain complex $\operatorname{colim} A_{kc}^{p,*}(X, \mathfrak{U}; V)$ is trivial by Lemma 5.15. By Proposition 5.13 the inclusion $A_{kc}^*(X; V)^G \hookrightarrow A_{kcg}^*(X; V)^G$ then induces an isomorphism in cohomology. □

Corollary 5.17. *For metrisable, locally compact or Hausdorff k_ω topological groups G which are contractible the continuous group cohomology $H_{kc,eq}(G; V)$ is isomorphic to the cohomology $H_{kcg,eq}(G; V)$ of homogeneous group cochains with continuous germ at the diagonal.*

6. COMPLEXES OF SMOOTH COCHAINS

In this Section we introduce the sub (double)complexes for smooth transformation groups (G, M) and smooth G -modules V , where V is an abelian Lie group. (We use the general infinite dimensional calculus introduced in [BGN04].) Let (G, M) be a smooth transformation group, V be a smooth G -module and \mathfrak{U} be an open covering of M .

Definition 6.1. For every manifold M and abelian Lie group V the subcomplex $A_s^*(M; V) := C^\infty(M^{*+1}; V)$ of the standard complex is called the *smooth standard complex*. The cohomology $H_{eq,s}(M; V)$ of the subcomplex $A_s^*(M; V)^G$ is called the equivariant smooth cohomology of M (with values in V).

Example 6.2. For any Lie group G which acts on itself by left translation and smooth G -module V the complex $A_s^*(G; V)^G$ is the complex of smooth (homogeneous) group cochains; its cohomology $H_{eq,s}(G; V)$ is the smooth group cohomology of G with values in V .

For Lie groups G and G -modules V the first cohomology group $H_{eq,s}^1(G; V)$ classifies smooth crossed morphisms modulo principal derivations, the second cohomology group $H_{eq,s}^2(G; V)$ classifies equivalence classes of Lie group extensions $V \hookrightarrow \hat{G} \twoheadrightarrow G$ which admit a smooth global section (i.e. $\hat{G} \twoheadrightarrow G$ is a trivial smooth V -principal bundle) and the third cohomology group $H_{eq,c}^3(G; V)$ classifies equivalence classes of smoothly split crossed modules.

For each open covering \mathfrak{U} of M one can consider the subcomplex of $A^*(M; V)$ formed by the groups

$$A_{sr}^n(M, \mathfrak{U}; V) := \{f \in A^n(M; V) \mid f|_{\mathfrak{U}[n]} \in C^\infty(\mathfrak{U}[n]; V)\}$$

of cochains whose restriction to the subspaces $\mathfrak{U}[n]$ of M^{n+1} are smooth. The cohomology of the cochain complex $A_{sr}^*(M, \mathfrak{U}; V)$ is denoted by $H_{sr}(M, \mathfrak{U}; V)$. If the covering \mathfrak{U} of M is G -invariant, then the subspaces $\mathfrak{U}[*]$ is a simplicial G -subspace of the simplicial G -space M^{*+1} . For G -invariant coverings \mathfrak{U} of M the cohomology of the subcomplex $A_{sr}^*(M, \mathfrak{U}; V)^G$ of G -equivariant cochains is denoted by $H_{cr,eq}(M, \mathfrak{U}; V)$.

Example 6.3. If $G = M$ is a Lie group which acts on itself by left translation and U an open identity neighbourhood, then the complex $A_{sr}^*(M, \mathfrak{U}_U; V)^G$ is the complex of homogeneous group cochains whose restrictions to the subspaces $\mathfrak{U}_U[*]$ are smooth. (These are sometimes called \mathfrak{U} -smooth cochains.)

For directed systems $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of M one can also consider the colimit complex $\text{colim}_i A_{sr}^*(M, \mathfrak{U}_i; V)$. In particular for the directed system of all open coverings of M one observes that the open diagonal neighbourhoods $\mathfrak{U}[n]$ in M^{n+1} for open coverings \mathfrak{U} of M are cofinal in the directed set of all open diagonal neighbourhoods, hence one obtains the complex

$$A_{sg}^*(M; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } M} A_{sr}^*(M; \mathfrak{U}; V)$$

of global cochains whose germs at the diagonal are continuous. This is a subcomplex of the standard complex $A^*(M; V)$ which is invariant under the G -action (Eq. 1.1) and thus a sub complex of G -modules. The G -equivariant cochains with continuous germ form a subcomplex $A_{sg}^*(M; V)^G$ thereof, whose cohomology is denoted by $H_{cg,eq}(M; V)$. The latter subcomplex can also be obtained by taking the colimit over all G -invariant open coverings of M only:

Proposition 6.4. *The natural morphism of cochain complexes*

$A_{cg,eq}^*(M; V) := \operatorname{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } M} A_{sr}^*(M; \mathfrak{U}; V)^G \rightarrow A_{sg}^*(M; V)^G$
is a natural isomorphism.

Proof. The proof is analogous to that of Proposition 1.7. \square

Corollary 6.5. *The cohomology $H_{cg,eq}(M; V)$ is the cohomology of the complex of equivariant cochains which are continuous on some G -invariant neighbourhood of the diagonal.*

Example 6.6. If $G = M$ is a Lie group which acts on itself by left translation, then the complex $A_{sg}^*(G; V)^G$ is the complex of homogeneous group cochains whose germs at the diagonal are smooth. (By abuse of language these are sometimes called 'locally smooth' group cochains.)

We will show (in Section 7) that the inclusion $A_{sr}^*(M, \mathfrak{U}; V) \hookrightarrow A_s^*(M; V)$ induces an isomorphism in cohomology provided the manifold M is smoothly contractible. For this purpose we consider the abelian groups

$$(6.1) \quad A_{sr}^{p,q}(M, \mathfrak{U}; V) := \{f : M^{p+1} \times M^{q+1} \rightarrow V \mid f|_{M^{p+1} \times \mathfrak{U}[q]} \text{ is continuous}\}.$$

The abelian groups $A_{sr}^{p,q}(M, \mathfrak{U}; V)$ form a first quadrant sub double complex of the double complex $A_{cr}^{p,q}(M, \mathfrak{U}; V)$. The rows of the double complex $A_{sr}^{*,*}(M, \mathfrak{U}; V)$ can be augmented by the complex $A_{sr}^*(M, \mathfrak{U}; V)$ for the covering \mathfrak{U} and the columns can be augmented by the exact complex $A_s^*(M; V)$ of continuous cochains:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\ A_{sr}^2(M, \mathfrak{U}; V) & \longrightarrow & A_{sr}^{0,2}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{1,2}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{2,2}(M, \mathfrak{U}; V) & \xrightarrow{d_h} \cdots \\ & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\ A_{sr}^1(M, \mathfrak{U}; V) & \longrightarrow & A_{sr}^{0,1}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{1,1}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{2,1}(M, \mathfrak{U}; V) & \xrightarrow{d_h} \cdots \\ & \uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\ A_{sr}^0(M, \mathfrak{U}; V) & \longrightarrow & A_{sr}^{0,0}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{1,0}(M, \mathfrak{U}; V) & \xrightarrow{d_h} & A_{sr}^{2,0}(M, \mathfrak{U}; V) & \xrightarrow{d_h} \cdots \\ & & \uparrow & & \uparrow & & \uparrow & \\ & & A_s^0(M; V) & \xrightarrow{d_h} & A_s^1(M; V) & \xrightarrow{d_h} & A_s^2(M; V) & \xrightarrow{d_h} \cdots \end{array}$$

We denote the total complex of the double complex $A_{sr}^{*,*}(M, \mathfrak{U}; V)$ by $\operatorname{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)$. The augmentations of the rows and columns of this double complex induce morphisms $i^* : A_{sr}^*(M, \mathfrak{U}; V) \rightarrow \operatorname{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)$ and $j^* : A_s^*(M; V) \rightarrow \operatorname{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)$ of cochain complexes respectively.

Lemma 6.7. *The morphism $i^* : A_{sr}^*(M, \mathfrak{U}; V) \rightarrow \operatorname{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)$ induces an isomorphism in cohomology.*

Proof. The row contraction given in the proof of Lemma 2.1 restricts to one of the sub row complex $A_{sr}^*(M, \mathfrak{U}; V) \hookrightarrow A_{sr}^{*,*}(M, \mathfrak{U}; V)$. \square

Remark 6.8. Note that this construction does not work for the column complexes.

For G -invariant open coverings \mathfrak{U} of M one can consider the sub double complex $A_{sr}^{*,*}(M, \mathfrak{U}; V)^G$ of $A_{sr}^{*,*}(M, \mathfrak{U}; V)$ whose rows are augmented by the cochain complex $A_{sr}^*(M, \mathfrak{U}; V)^G$ for the covering \mathfrak{U} and the columns can be augmented by the complex $A_s^*(M; V)^G$ of smooth equivariant cochains (which is not exact in general).

Lemma 6.9. *For G -invariant coverings \mathfrak{U} of M the morphism $i_{eq}^* := i^*{}^G$ induces an isomorphism in cohomology.*

Proof. The contraction $h_{*,q}$ of the augmented rows $A_{cr}^q(M, \mathfrak{U}; V) \hookrightarrow \text{Tot} A_{sr}^{*,q}(M, \mathfrak{U}; V)$ defined in Eq. 2.2 is G -equivariant and thus restricts to a row contraction of the augmented sub-row $A_{sr}^q(M, \mathfrak{U}; V)^G \hookrightarrow \text{Tot} A_{sr}^{*,q}(M, \mathfrak{U}; V)^G$. \square

So the morphism $H(i_{eq}) : H_{cr,eq}(M, \mathfrak{U}; V) \rightarrow H(\text{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)^G)$ is invertible. For the composition $H(i_{eq})^{-1}H(j_{eq}) : H_{c,eq}(M; V) \rightarrow H_{cr,eq}(M, \mathfrak{U}; V)$ we observe:

Proposition 6.10. *The image $j^n(f)$ of a smooth equivariant n -cocycle f on M in $\text{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)^G$ is cohomologous to the image $i_{eq}^n(f)$ of the equivariant n -cocycle $f \in A_{sr}^n(M, \mathfrak{U}; V)^G$ in $\text{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)^G$.*

Proof. The proof is analogous to that of Proposition 2.4. \square

Corollary 6.11. *The composition $H(i_{eq})^{-1}H(j_{eq}) : H_{s,eq}(M; V) \rightarrow H_{sr,eq}(M, \mathfrak{U}; V)$ is induced by the inclusion $A_s^*(M, \mathfrak{U}; V)^G \hookrightarrow A_{sr}^*(M, \mathfrak{U}; V)^G$.*

Corollary 6.12. *If the morphism $j_{eq}^* := j^*{}^G : A_s^*(M; V)^G \rightarrow \text{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sr}^*(M, \mathfrak{U}; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of M one can also consider the corresponding augmented colimit double complexes. In particular for the directed system of all open coverings of M one obtains the double complex complex

$$A_{sg}^{*,*}(M; V) := \text{colim}_{\mathfrak{U} \text{ is open cover of } M} A_{sr}^{*,*}(M, \mathfrak{U}; V)$$

whose rows and columns are augmented by the colimit complex $A_{sg}^*(M; V)$ and by the complex $A_s^*(M; V)$ respectively.

Lemma 6.13. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of open coverings of M the morphism $\text{colim}_i i^* : \text{colim}_i A_{sr}^*(M, \mathfrak{U}_i; V) \rightarrow \text{Tot colim}_i A_{sr}^{*,*}(M, \mathfrak{U}_i; V)$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 6.7). \square

As a consequence the colimit morphism $i_{sg}^* : A_{sg}^*(M; V) \rightarrow \text{Tot} A_{sg}^{*,*}(M; V)$ induces an isomorphism in cohomology. The colimit double complex $A_{sg}^{*,*}(M; V)$ is a double complex of G -modules and the G -equivariant cochains in form a sub double complex $A_{sg}^{*,*}(M; V)^G$, whose rows and columns are augmented by the colimit complex $A_{cg,eq}^*(M; V)$ and by the complex $A_s^*(M; V)^G$ respectively.

Lemma 6.14. *For any directed system $\{\mathfrak{U}_i \mid i \in I\}$ of G -invariant open coverings of M the morphism $\text{colim}_i i_{eq}^* : \text{colim}_i A_{sr}^*(M, \mathfrak{U}_i; V)^G \rightarrow \text{Tot colim}_i A_{sr}^{*,*}(M, \mathfrak{U}_i; V)^G$ induces an isomorphism in cohomology.*

Proof. The passage to the colimit preserves the exactness of the augmented row complexes (Lemma 6.9). \square

Moreover, since the open diagonal neighbourhoods $\mathfrak{U}[n]$ in X^{n+1} for open coverings \mathfrak{U} of X are cofinal in the directed set of all open diagonal neighbourhoods, we observe:

Lemma 6.15. *The natural morphism of double complexes*

$$A_{cg,eq}^{*,*}(X; V) := \operatorname{colim}_{\mathfrak{U} \text{ is } G\text{-invariant open cover of } X} A_{sr}^{*,*}(X; \mathfrak{U}; V)^G \rightarrow A_{sg}^*(X; V)^G$$

is a natural isomorphism.

Proof. The proof is analogous to that of Proposition 1.7. \square

As a consequence the colimit morphism $i_{cg,eq}^* : A_{cg,eq}^*(M; V) \rightarrow \operatorname{Tot} A_{sg}^{*,*}(M; V)^G$ induces an isomorphism in cohomology, and the morphism $H(i_{cg,eq})$ is invertible. For the composition $H(i_{cg,eq})^{-1} H(j_{eq}) : H_{c,eq}(M; V) \rightarrow H_{cg,eq}(M, \mathfrak{U}; V)$ we observe:

Proposition 6.16. *The image $j^n(f)$ of a continuous equivariant n -cocycle f on M in $\operatorname{Tot} A_{sg}^{*,*}(M; , V)^G$ is cohomologous to the image $i_{cg,eq}^n(f)$ of the equivariant n -cocycle $f \in A_{cg,eq}^n(M; V)$ in $\operatorname{Tot} A_{sg}^{*,*}(M; V)^G$.*

Proof. The proof is analogous to that of Proposition 2.4. \square

Corollary 6.17. *The composition $H(i_{cg,eq})^{-1} H(j_{eq}) : H_{c,eq}(M; V) \rightarrow H_{cg,eq}(M; V)$ is induced by the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$.*

Corollary 6.18. *If the morphism $j_{eq}^* := j^{*G} : A_s^*(M; V)^G \rightarrow \operatorname{Tot} A_{sg}^{*,*}(M; V)^G$ induces a monomorphism, epimorphism or isomorphism in cohomology, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{cg,eq}^*(M; V)$ induces a monomorphism, epimorphism or isomorphism in cohomology respectively.*

7. SMOOTH AND \mathfrak{U} -SMOOTH COCHAINS

In this Section we derive results for smooth transformation groups (G, M) and smooth G -modules V , which are analogous to those concerning continuous cochains. Let (G, M) be a smooth transformation group, V be a smooth G -module and \mathfrak{U} be an open covering of M .

Proposition 7.1. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sr}^{p,*}(M, \mathfrak{U}; V)$ are exact, then the augmented sub column complexes $A_s^p(M; V)^G \hookrightarrow A_{sr}^{p,*}(M, \mathfrak{U}; V)^G$ of equivariant cochains are exact as well.*

Proof. The proof is analogous to that of Proposition 3.1. \square

Corollary 7.2. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sr}^{p,*}(M, \mathfrak{U}; V)$ are exact, then the inclusion $j_{eq}^* : A_s^*(M; V)^G \hookrightarrow \operatorname{Tot} A_{sr}^{*,*}(M, \mathfrak{U}, V)^G$ induces an isomorphism in cohomology.*

Corollary 7.3. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sr}^{p,*}(M, \mathfrak{U}; V)$ are exact, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sr}^*(M, \mathfrak{U}; V)^G$ induces an isomorphism in cohomology.*

It remains to show that for smoothly contractible manifolds M the colimit augmented columns $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V)$ are exact. For this purpose we first consider the cochain complex associated to the cosimplicial abelian group $A^{p,*}(M; V) := \{f : M^{p+1} \times M^{*+1} \rightarrow V \mid \forall \vec{x}' \in M^{*+1} : f(-, \vec{m}') \in C^\infty(M^{p+1}, V)\}$ of global cochains, its subcomplex $A_{sr}^{p,*}(M, \mathfrak{U}; V)$ and the cochain complexes associated to the cosimplicial abelian groups

$$\begin{aligned} A^{p,*}(\mathfrak{U}; V) &:= \{f : M^{p+1} \times \mathfrak{U}[*] \rightarrow V \mid \forall \vec{m}' \in \mathfrak{U}[*] : f(-, \vec{m}') \in C^\infty(M^{p+1}, V)\} \quad \text{and} \\ A_s^{p,*}(M, \mathfrak{U}; V) &:= C^\infty(M^{p+1} \times \mathfrak{U}[*], V). \end{aligned}$$

Restriction of global to local cochains induces morphisms of cochain complexes $\text{Res}^{p,*} : A^{p,*}(M; V) \rightarrow A^{p,*}(M, \mathfrak{U}; V)$ and $\text{Res}_{sr}^{p,*} : A_{sr}^{p,*}(M, \mathfrak{U}; V) \rightarrow A_s^{p,*}(M, \mathfrak{U}; V)$ intertwining the inclusions of the subcomplexes $A_{sr}^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A^{p,*}(M; V)$ and $A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A^{p,*}(M, \mathfrak{U}; V)$, so one obtains the following commutative diagram

$$(7.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker(\text{Res}_{sr}^{p,*}) & \longrightarrow & A_{sr}^{p,*}(M, \mathfrak{U}; V) & \longrightarrow & A_s^{p,*}(M, \mathfrak{U}; V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(\text{Res}^{p,*}) & \longrightarrow & A^{p,*}(M; V) & \longrightarrow & A^{p,*}(M, \mathfrak{U}; V) \longrightarrow 0 \end{array}$$

of cochain complexes whose rows are exact. The kernel $\ker(\text{Res}^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $M^{p+1} \times \mathfrak{U}[q]$. Since these (p, q) -cochains are smooth on $M^{p+1} \times \mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}^{p,*}) = \ker(\text{Res}_{rc}^{p,*})$ by $K^{p,*}$ and denote the cohomology groups of the complex $A_{sr}^{p,*}(M, \mathfrak{U}; V)$ by $H_{sr}^{p,*}(M, \mathfrak{U}; V)$, the cohomology groups of the complex $A_s^{p,*}(M, \mathfrak{U}; V)$ of continuous cochains by $H_s^{p,*}(M, \mathfrak{U}; V)$ and the cohomology groups of the complex $A^{p,*}(M, \mathfrak{U}; V)$ by $H^{p,*}(M, \mathfrak{U}; V)$.

Lemma 7.4. *The cochain complexes $A^{p,*}(M; V)$ are exact.*

Proof. For any point $* \in M$ the homomorphisms $h^{p,q} : A^{p,q}(M; V) \rightarrow A^{p,q-1}(M; V)$ given by $h^{p,q}(f)(\vec{x}, \vec{x}') := f(\vec{x}, *, \vec{x}')$ form a contraction of the complex $A^{p,*}(M; V)$. \square

The morphism of short exact sequences of cochain complexes in diagram 3.1 gives rise to a morphism of long exact cohomology sequences, in which the cohomology of the complex $A^{p,*}(M; V)$ is trivial:

$$(7.2) \quad \begin{array}{ccccccc} \longrightarrow & H^q(K^{p,*}) & \longrightarrow & H_{sr}^{p,q}(M, \mathfrak{U}; V) & \longrightarrow & H_s^{p,q}(M, \mathfrak{U}; V) & \longrightarrow & H^{q+1}(K^{p,*}) \longrightarrow \\ & \parallel & & \downarrow & & \downarrow & & \parallel \\ \longrightarrow & \cong H^q(K^{p,*}) & \longrightarrow & 0 & \longrightarrow & H^{p,q}(M, \mathfrak{U}; V) & \xrightarrow{\cong} & H^{q+1}(K^{p,*}) \longrightarrow \end{array}$$

Lemma 7.5. *If the inclusion $A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A^{p,*}(M, \mathfrak{U}; V)$ induces an isomorphism in cohomology, then the augmented complex $A_s^p(M; V) \hookrightarrow A_{sr}^{p,*}(M, \mathfrak{U}; V)$ is exact.*

Proof. This is an immediate consequence of Diagram 7.2 \square

Proposition 7.6. *If the inclusion $A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A^{p,*}(M, \mathfrak{U}; V)$ induces an isomorphism in cohomology, then the inclusions $j_{eq}^* : A_s^*(M; V)^G \hookrightarrow \text{Tot} A_{sr}^{*,*}(M, \mathfrak{U}; V)^G$ and $A_s^*(M, \mathfrak{U}; V)^G \hookrightarrow A_{sr}^*(M, \mathfrak{U}; V)^G$ also induces an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 7.2 and 7.3. \square

The passage to the colimit over all open coverings of M yields the corresponding results for the complexes of cochains with continuous germs:

Proposition 7.7. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V)$ are exact, then the augmented sub column complexes $A_s^p(M; V)^G \hookrightarrow A_{sg}^{p,*}(M; V)^G$ of equivariant cochains are exact as well.*

Proof. The proof is similar to that of Proposition 3.1. \square

Corollary 7.8. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V)$ are exact, then the inclusion $j_{eq}^* : A_s^*(M; V)^G \hookrightarrow \text{Tot} A_{sg}^{*,*}(M; V)^G$ induces an isomorphism in cohomology.*

Corollary 7.9. *If the augmented column complexes $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V)$ are exact, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$ induces an isomorphism in cohomology.*

Remark 7.10. Alternatively to taking the colimit over all open coverings \mathfrak{U} of M one may consider G -invariant open coverings only to obtain the same results. (This was shown in Proposition 6.4 and Lemmata 6.15.)

Example 7.11. If $G = M$ is a Lie group which acts on itself by left translation and the augmented columns $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V) := \text{colim } A^{p,*}(M, \mathfrak{U}_U; V)$ (where U runs over all open identity neighbourhoods in G) are exact, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$ induces an isomorphism in cohomology.

The complex $A^{p,*}(M, \mathfrak{U}; V)$ is isomorphic to the complex $A^*(\mathfrak{U}; C(M^{p+1}, V))$. The colimit $A_{AS}^*(M; C(M^{p+1}, V)) := \text{colim } A^*(\mathfrak{U}; C(M^{p+1}, V))$, where \mathfrak{U} runs over all open coverings of M is the complex of Alexander-Spanier cochains on M . Therefore the colimit complex $\text{colim } A^p(M; A^*(\mathfrak{U}; V))$ is isomorphic to the cochain complex $A_{AS}^*(M; C(M^{p+1}, V))$. A similar observation can be made for the cochain complex $A_{sg}^{p,*}(M, \mathfrak{U}; V)$ if the exponential law $C(M^{p+1} \times \mathfrak{U}[q], V) \cong C(M, C(\mathfrak{U}[q], V))$ holds for a cofinal set of open coverings \mathfrak{U} of M . Passing to the colimit in Diagram 7.1 yields the morphism

$$(7.3) \quad \begin{array}{ccccccc} 0 \longrightarrow & \ker(\text{Res}_{sg}^{p,*}) & \longrightarrow & A_{sg}^{p,*}(M; V) & \longrightarrow & \text{colim } A_{sg}^{p,*}(M, \mathfrak{U}; V) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \ker(\text{Res}^{p,*}) & \longrightarrow & A^{p,*}(M; V) & \longrightarrow & A_{AS}^*(M; C^{p+1}(M, V)) & \longrightarrow 0 \end{array}$$

of short exact sequences of cochain complexes. The kernel $\ker(\text{Res}^{p,q})$ is the subspace of those (p, q) -cochains which are trivial on $M^{p+1} \times \mathfrak{U}[q]$ for some open covering \mathfrak{U} of M . Since these (p, q) -cochains are continuous on $M^{p+1} \times \mathfrak{U}[q]$ we find that both kernels coincide. We abbreviate the complex $\ker(\text{Res}^{p,*}) = \ker(\text{Res}_{sg}^{p,*})$ by $K_{sg}^{p,*}$ and denote the cohomology groups of the complex $A_{sg}^{p,*}(M; V)$ by $H_{sg}^{p,*}(M; V)$. The morphism of short exact sequences of cochain complexes in Diagram 7.3 gives rise to a morphism of long exact cohomology sequences:

$$(7.4) \quad \begin{array}{ccccccc} \longrightarrow & H^q(K_{sg}^{p,*}) & \longrightarrow & H_{sg}^{p,q}(M, \mathfrak{U}; V) & \longrightarrow & H^q(\text{colim } A_{sg}^{p,*}(M, \mathfrak{U}; V)) & \longrightarrow & H^{q+1}(K_{sg}^{p,*}) & \longrightarrow \\ & \parallel & & \downarrow & & \downarrow & & \parallel & \\ \longrightarrow & \cong H^q(K^{p,*}) & \longrightarrow & 0 & \longrightarrow & H_{AS}^q(M; C^{p+1}(M, V)) & \xrightarrow{\cong} & H^{q+1}(K^{p,*}) & \longrightarrow \end{array}$$

Lemma 7.12. *If the inclusion $\operatorname{colim} A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(M; C^{p+1}(M, V))$ of cochain complexes induces an isomorphism in cohomology, then the augmented complex $A_s^p(M; V) \hookrightarrow A_{sg}^{p,*}(M; V)$ is exact.*

Proof. This is an immediate consequence of Diagram 7.4 \square

Proposition 7.13. *If the inclusion $\operatorname{colim} A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(M; C(M^{p+1}, V))$ induces an isomorphism in cohomology, then $j_{eq}^* : A_s^*(M; V)^G \hookrightarrow \operatorname{Tot} A_{sg}^{*,*}(M; V)^G$ and $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$ also induce an isomorphism in cohomology.*

Proof. This follows from the preceding Lemma and Corollaries 7.8 and 7.9. \square

As observed before (cf. Remark 7.10) one may restrict oneself to the directed system of G -invariant open coverings only to achieve the same result. Thus we observe:

Corollary 7.14. *If $G = M$ is a Lie group which acts on itself by left translation and the inclusion $\operatorname{colim} A_s^{p,*}(M, \mathfrak{U}; V) \hookrightarrow A_{AS}^*(M; C(M^{p+1}, V))$ (where U runs over all open identity neighbourhoods in G) induces an isomorphism in cohomology, then the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$ induces an isomorphism in cohomology as well.*

Proof. It has been shown in [vE62] that the cohomology of the colimit cochain complex $\operatorname{colim} A^*(\mathfrak{U}; C(M^{p+1}, V))$ is the Alexander-Spanier cohomology of M . \square

Lemma 7.15. *If the manifold M is contractible, then the cohomology of the complex $\operatorname{colim} A_s^{p,*}(M, \mathfrak{U}; V)$ is trivial.*

Proof. The reasoning is analogous to that for the Alexander-Spanier presheaf. The proof [Fuc10, Theorem 2.5.2] carries over almost in verbatim. \square

Proof. If the manifold M is contractible, then the Alexander-Spanier cohomology $H_{AS}(M; C^{p+1}(M, V))$ is trivial and the cohomology of the cochain complex $\operatorname{colim} A_s^{p,*}(M, \mathfrak{U}; V)$ is trivial by Lemma 7.15. By Proposition 7.13 the inclusion $A_s^*(M; V)^G \hookrightarrow A_{sg}^*(M; V)^G$ then induces an isomorphism in cohomology. \square

Corollary 7.16. *For smoothly contractible Lie groups G the continuous group cohomology is isomorphic to the cohomology of homogeneous group cochains with continuous germ at the diagonal.*

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E-mail address: martin@fuchssteiner.net